3. Gabasov, R. and Kirillova, F., Qualitative Theory of Optimal Processes. Moscow, "Nauka", 1971.
4. Lee, E. B. and Markus, L., Foundations of the Theory of Optimal Control. Moscow, "Nauka", 1971.
5. Lukes, D. L., Global controllability of nonlinear systems. SJAM J. Control., Vol. 10, №1, 1972.
6. Vidyasagar, M., A controllability condition for nonlinear systems. IEEE Trans. Automat. Control., Vol. 17, №4, 1972.
7. Mirza, K. B. and Womack, B., On the controllability of a class of nonlinear systems. IEEE Trans. Automat. Control, Vol. 17, №4, 1972.
8. Dauer, J. P., Sufficient conditions for controllability of nonlinear systems. Atti Accad. naz. Lincei Rend. cl. sci fis., mat. e. natur., Vol. 51, №5, 1971 (1972).
9. Dauer, J. P., A controllability technique for nonlinear systems. J. Math. Analysis and Applic., Vol. 37, №2, 1972.
10. Aronsson, G., A new approach to nonlinear controllability. J. Math. Analysis and Applic., Vol. 44, №3, 1973.

Translated by N. H. C.

UDC 531.36

STABILITY ANALYSIS OF DYNAMIC SYSTEMS WITH COUPLINGS AND INTEGRALS OF MOTIONS

PMM Vol. 38, №4, 1974, pp. 607-615
P. WILLEMS
(Belgium)
(Received March 6, 1973)

We give a method for obtaining the stability conditions for nonlinear systems, based on an analysis of the linearized coupling equations and of the linearized or quadratic expressions for the integrals of motion. Liapunov's method is usually employed in the investigation of the stability of dynamic systems. The investigation of the Hamiltonian function is a convenient tool for systems with internal energy dissipation. In fact, in the development of the Thompson (Lord Kelvin)-Tait-Chetaev theorem [1 – 4] it was shown that the positive definiteness of the Hamiltonian function provides the necessary and sufficient stability conditions in the case of complete dissipation. We have obtained just sufficient conditions for system with partial dissipation; moreover, the method does not yield the possibility of obtaining far-reaching inferences on stability on the basis of the analysis of the linearized equations. It should be noted also that in several cases it is convenient to introduce a number of variables, exceeding the number of degrees of freedom, and to examine the couplings. Then the equations can be simplified or represented in a form convenient for stability analysis.

We can derive the equations of motion of the system under consideration with the aid of the Lagrange function of the form

$$L = {}^1/_2 q^{\cdot T} M' q^{\cdot} + q^{\cdot T} \Gamma q + {}^1/_2 q^T K' q \tag{1}$$

Here q is the $m \times 1$ generalized coordinate vector, M' and K' are symmetric matrices. The matrices M', K' and Γ are functions of q which, by definition, are polynomials. The superscript T denotes transposition. If the system is nonholonomic or is described by a larger number of variables than the number of degrees of freedom , we introduce the coupling equations. We assume that no work is accomplished on the couplings and that they are expressed in the form of kinematic relations

$$A^T dq = 0 \tag{2}$$

If n is the number of degrees of freedom, matrix A is of dimension $n \times (m - n)$ and is a function (a polynomial) of m variables.

It is well known that under the assumptions made, the equations of motion in a neighborhood of the equilibrium $q = 0$ form a system of $2m - n$ equations

$$M' q^{\cdot\cdot} + G' q^{\cdot} + K' q - A\lambda = Q', \qquad A^T q^{\cdot} = 0 \tag{3}$$

Here $G' = \Gamma - \Gamma^T$ is a skew-symmetric matrix, Q' is the matrix of generalized forces, including the dissipation forces, and equals zero when $q = 0$. Let us show that the stability conditions for system (3) in substance coincide with the stability conditions for the equivalent linear system with integrable couplings.

Liapunov's theorem on stability in the first approximation cannot be applied directly for determining the stability of system (3) near the equilibrium position $q = 0$ because the corresponding linear system has zero eigenvalues. In such a case the stability can depend upon nonlinear terms.

A linear holonomic system, equivalent to linearized system (3), can be obtained by Whittaker's method [5]. The integral of the linearized $A^T q^{\cdot} = 0$ has the form

$$A^T q = 0 \tag{4}$$

and $q = 0$ is the solution. If

$$A^T = A_0 B, \qquad q = [x_0{}^T x^T]^T$$

where A_0 is an ($(m - n) \times (m - n)$)-matrix and x_0 is an ($(m - n) \times 1$)-vector, then the variable x_0 can be expressed in terms of x as

$$x_0 = Cx, \qquad C = - A_0^{-1} B \tag{5}$$

The required system is obtained by replacing x_0 by the value given by formula (5) in the expressions for the kinetic and potential energies.

If the matrices M', G' and K' are of the form

$$M' = \begin{Vmatrix} M^{\circ} & M^{\circ\alpha} \\ M^{\beta\circ} & M^{\beta\alpha} \end{Vmatrix}, \qquad G' = \begin{Vmatrix} G^{\circ} & G^{\circ\alpha} \\ G^{\beta\circ} & G^{\beta\alpha} \end{Vmatrix}$$

$$K' = \begin{Vmatrix} K^{\circ} & K^{\circ\alpha} \\ K^{\beta\circ} & K^{\beta\alpha} \end{Vmatrix}$$

then the corresponding holonomic system is

$$Mx^{\cdot\cdot} + Gx^{\cdot} + Kx = Q \tag{6}$$
$$(M = M^{\beta\alpha} + C^T M^{\circ} C + C^T M^{\circ\alpha} + M^{\beta\circ} C)$$

where Q are the generalized forces coupled with variable x. Expressions analogous to those within the parantheses hold also for G and K. It should be noted that we can obtain system (6) directly by multiplying the equations of system (3) by $C^T E$, where E is the $(m - n) \times (m - n)$ unit matrix and by replacing x_0 by expression (5). With the aid of the Hamiltonian function and of relation (5), considered as an integral of motion, we can show that the stability of linearized system (3) is equivalent to the stability of the corresponding system (6).

Liapunov's theorem on stability in the first approximation is applicable in the case of nonlinear systems and of integrable coupling equations with due regard to the linearized integral (5). However, the couplings can be unintegrable in the case of a nonholonomic system, and merely the boundedness of (3) follows from the asymptotic stability of (6). If system (6) has one eigenvalue with a positive real part, then the linear system (3) is unstable in all the cases indicated above.

It is clear that such a procedure can be extended to systems of equations which are not derived directly from the Lagrange function. It is usually simpler to obtain the Euler-Liouville equations [6 – 8] in the case of a rotational motion because here we are not required to carry out a quadratic approximation for the transformation matrices and for the expressions for angular velocities, as is required by the Lagrange method [9]. The stated equations are obtained in the form

$$T\,(Mx^{\cdot\cdot} + Gx + Kx - Q) = 0 \tag{7}$$

It is difficult to determine the matrix T in the general case; therefore, we cannot call on the Hamiltonian function to make a judgement on the stability of (7). However, we can sometimes write a system of variables whose number exceeds the number of degrees of freedom and make use of couplings; this allows us to obtain a system of equations directly in the form (6).

The Hamiltonian function of system (6) is

$$H = {}^1/_2\,(x^{\cdot T} M x^{\cdot} + x^T K x)$$

and in the presence of dissipation its time derivative along a trajectory is $H^{\cdot} = x^{\cdot T} Q \leqslant 0$ everywhere in the state space. If H is a positive-definite function and if $H^{\cdot} \neq 0$ along a trajectory (the case of complete dissipation), then the trivial solution is asymptotically stable. On the other hand, if function H can take negative values in the neighborhood of the origin and if the dissipation is complete, the system is unstable. If $H^{\cdot} \equiv 0$ along a trajectory in state space, then the positive definiteness of H provides only sufficient conditions for stability but not for asymptotic stability.

The generalized mass matrix M is always positive and is positive definite for the majority of cases considered in problems of the dynamics of rotational motion. In this case H is positive definite if matrix K possesses this property. The positive definiteness of K can be verified by the Sylvester conditions; this yields n stability conditions.

Some systems do not possess complete dissipation, and an arbitrary choice of initial conditions can alter certain integrals of motion of the system. For example, a system with free rotation, when the initial conditions can change the total angular momentum, is such a system. In the presence of dissipation the system tends to a new equilibrium position.

If a system has integrals of motion, several methods can be used to settle the question of its stability or partial stability when the integrals are satisfied exactly or approximately. We do not examine these methods here and propose a simple way of achieving the results needed for solving the particular problem being considered. When the integrals are satisfied approximately, we can make a judgement only on nonasymptotic stability [10]. Therefore, let us examine systems for which the integrals are satisfied and investigate the system's stability when the trajectory lies wholly on a hypersurface defined by the integrals of motion. Here we also assume that the integrals of motion can be described by polynomials.

We introduce a new function, being a combination of the Hamiltonian function and of the integrals of motion, so that certain variables are eliminated. The new Liapunov function is expressed in terms of the remaining variables x'. Its time derivative under initial conditions satisfying the integrals of motion equals $H^{\cdot}$, and the dissipation now can be complete. The positive definiteness of the polynomial relative to the zero value is determined by the quadratic terms; therefore, the system is asymptotically stable with respect to a part of the variables (m variables x') if the new function $V = x'^T K' x'$ is positive definite. This leads to a system of m stability conditions. For many problems it can be shown that asymptotic stability follows from this stability with respect to a part of variables x'.

Stability of the regular precession of a heavy gyroscope mounted on gimbals. The exact solution of the problem for the case of the gyro rotation around a vertical axis of the outer gimbal was given by Magnus [11]; The stability conditions for all equilibrium positions were given by Rumiantsev [12] (see also [13] for the results). We take the inertia of the outer gimbal into account and we reckon that the dissipation is caused by the inner gimbal. The outer gimbal rotation axis is vertical. The center of mass of the system consisting of the gyro and the inner gimbal is located on the gyro axis of symmetry. The center of mass of the outer gimbal coincides with the point O of intersection of the rotation axes of both gimbals.

We introduce the body coordinate systems $OX_1^*X_2^*X_3^*$ and $Ox_1^*x_2^*x_3^*$, for the housing and for the gyro, respectively. The outer gimbal rotation axis is directed along the X_3^*, the inner gimbal — along x_1^*, and the gyro symmetry axis — along x_3^*. After the system $OX_1^*X_2^*X_3^*$ rotates around the axis X_3^* by an angle φ the axes X_α^* coincide with the axes of the outer gimbal; analogously, the axes of the outer gimbal coincide with the axes x_α^* after a rotation by angle θ around the axis X_1^* (or x_1^*). We denote by $\psi^{\cdot}$ the angular velocity of the gyro relative to the inner gimbal. The position of the center of mass relative to point O is determined by a vector of length d directed along the axis x_3^*.

We introduce the dimensionless parameters

$$k_1 = \frac{I}{J_1}, \qquad k_2 = \frac{mgd}{J_1}, \qquad k_3 = \frac{A + J_2}{J_1}$$

$$k_4 = \frac{J_2 + I - J_3}{J_1}, \qquad k_5 = \frac{J_2 - J_3}{J_1}$$

Here I is the gyro moment of inertia relative to its symmetry axis. J_α are the gyro moments of inertia relative to axes x_α^*, A is the moment of inertia of the outer gimbal relative to its rotation axis, m is the mass of the gyro, g is the gravitational acceleration.

The equations of motion are of the form

$$\theta^{\cdot\cdot} - k_5\Phi^{\cdot 2}\sin\theta\cos\theta + k_1\psi^{\cdot}\Phi^{\cdot}\sin\theta - k_2\sin\theta = 0 \tag{8}$$

$$\frac{d}{dt}[(k_3 - k_5\cos^2\theta)\,\Phi^{\cdot} + k_1\psi^{\cdot}\cos\theta] = 0$$

$$\frac{d}{dt}[k_1(\psi^{\cdot} + \Phi^{\cdot}\cos\theta)] = L$$

Here L is the force moment of the inner rotation around the gyro axis. We assume that this moment equals zero for some values of $\psi^{\cdot}$. The system has several equilibrium positions; but here we examine only the case of regular precession when

$$\theta = \theta_0,\ \Phi^{\cdot} = \Phi_0^{\cdot},\ \psi^{\cdot} = \psi_0^{\cdot},\ \theta^{\cdot} = \Phi^{\cdot\cdot} = \Phi^{\cdot\cdot} = 0$$

In this case, allowing that $\theta_0 \neq 0$ for a regular precession, from the first equation in (8) we obtain

$$-k_5\Phi_0^{\cdot 2}\cos\theta_0 + k_1\psi_0^{\cdot}\,\Phi_0^{\cdot} - k_2 = 0 \tag{9}$$

For the perturbed motion relative to equilibrium, determined by (9), we introduce the variables $x_1,\ x_2,\ x_3$ such that

$$\theta = \theta_0 + x_1, \qquad \theta^{\cdot} = x_1^{\cdot}$$
$$\Phi^{\cdot} = \Phi_0^{\cdot} + x_2^{\cdot}, \qquad \psi^{\cdot} = \psi_0^{\cdot} + x_3^{\cdot}$$

If dissipation is connected with the axis of the inner gimbal (c' is the dissipation factor) and if the quantity L has been linearized, $L / I = -c\psi^{\cdot} - k\psi$, then the equations of motion have a form analogous to (6), i.e.

$$Mx^{\cdot\cdot} + Gx^{\cdot} + Kx = -Dx^{\cdot}, \qquad x = -(x_1,\ x_2,\ x_3]^T \tag{10}$$

Here

$$D = \mathrm{diag}\,(c',\ 0,\ c), \qquad K = \mathrm{diag}\,(k_5\varphi_0^{\cdot 2}\sin\theta_0,\ 0,\ k)$$

$$M = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & k_3 - k_5\cos^2\theta_0 & k_1\cos\theta_0 \\ 0 & k_1\cos\theta_0 & k_1 \end{Vmatrix}, \quad G = \begin{Vmatrix} 0 & g_{12} & g_{13} \\ -g_{12} & 0 & 0 \\ -g_{13} & 0 & 0 \end{Vmatrix}$$

$$g_{12} = -2k_5\Phi_0^{\cdot}\sin\theta_0\cos\theta_0 + k_1\psi_0^{\cdot}\sin\theta_0$$
$$g_{13} = k_1\Phi_0^{\cdot}\sin\theta_0$$

Then the Hamiltonian function's quadratic form is

$$2H = x_1^{\cdot 2} + (k_3 - k_5\cos^2\theta_0)\,x_2^{\cdot 2} + k_1 x_3^{\cdot 2} + 2k_1\cos\theta_0 x_2^{\cdot}x_3^{\cdot} + \tag{11}$$
$$k_5\Phi_{01}^{\cdot 2}\sin^2\theta_0 x_1^2 + kx_3^2$$
$$H = -c'x_1^{\cdot 2} - cx_3^{\cdot 2}$$

It should be noted that dissipation cannot be connected with the variable φ because regular precession can be absent in this case.

The system has at least one integral of motion, which follows from the first equation in (8). This integral corresponds to the component of angular momentum relative to axis

$X_3{}^*$ (the moment of external forces relative to this axis are assumed absent). In linearized form the integral mentioned has the form

$$(k_3 - k_5 \cos^2 \theta_0)\, x_2{}^\bullet + k_1 \cos \theta_0\, x_3{}^\bullet + (2k_5\Phi_0{}^\bullet \sin \theta_0 \cos \theta_0 - k_1\psi_0{}^\bullet \sin \theta_0)\, x_1 = 0 \tag{12}$$

With the aid of (11) and (12) we can construct a function V which is quadratic in x_1, x_3, $x_1{}^\bullet$ and $x_3{}^\bullet$. For initial conditions satisfying (12), the dissipation is complete. The positive definiteness of V is a necessary and sufficient stability condition for system (10) and we can then obtain the stability condition for the regular precession (excepting for the critical cases). Simple conditions in closed form can be obtained in the limiting cases when k and c equal zero or infinity. In the first case the dissipation is compensated by the gyro drive; then the system has a second integral of motion whose linearized form is written as

$$k_1 x_3{}^\bullet + k_1 x_2{}^\bullet \cos \theta_0 - k_1 \Phi_0{}^\bullet \sin \theta_0 x_1 = 0 \tag{13}$$

The dissipation is complete and, from (12) and (13), allowing for (11), we construct a function V', quadratic in x_1 and $x_1{}^\bullet$, in the form

$$V' = x_1{}^{\bullet 2} + \left\{ k_4 \Phi_0{}^{\bullet 2} + \frac{(k_1(\psi_0{}^\bullet + \Phi_0{}^\bullet \cos \theta_0) - 2k_4\Phi_0{}^\bullet \cos \theta_0)^2}{k_3 - k_4 \cos^2 \theta_0} \right\} \sin^2 \theta_0 x_1{}^2$$

The time derivative of this function equals $-c' x_1{}^{\bullet 2}$.

System (10) with integrals of motion (12) and (13) is asymptotically stable if

$$k_4 \Phi_0{}^{\bullet 2} + \frac{(k_1(\psi_0{}^\bullet + \Phi_0{}^\bullet \cos \theta_0) - 2k_4\Phi_0{}^\bullet \cos \theta_0)^2}{k_3 - k_4 \cos^2 \theta_0} > 0 \tag{14}$$

This condition is usually satisfied for real systems since $k_4 > 0$ and $k_3 > k_4$.

On the basis of the results in [10] we see that condition (14) is sufficient for the stability of system (10) also when equalities (12) and (13) are satisfied approximately.

It should be noted that the existence condition for a regular precession must be satisfied. This follows from (9) in the form

$$k_1{}^2 \psi_0{}^2 \geqslant 4\, k_2 k_5 \cos \theta_0$$

This condition is always satisfied for high values of the gyro rotation velocity. Condition (14) can be obtained also by examining the nonlinear system and the positive definiteness of the Routh function or by examining a linear combination of the integrals of motion. It can be verified that function V' equals the doubled sum of the quadratic terms of the Routh function obtained in [12]. Therefore, the necessary stability condition is obtained by a direct application of the results of the paper cited. The sufficient condition, proposed in [12], is equivalent to $k_4 > 0$. It is clear that this condition is sufficient for the fulfillment of (14).

Stability of a system of freely-rotating intercoupled rigid bodies. We assume that the system indicated rotates at a constant angular velocity ω_0 around axis $X_3{}^*$. For each element of the system the Euler equations are written in the body coordinate system. If one of the bodies is taken as the reference origin, the total number of variables (the angular deformations in the joints of the bodies and the angular velocity of the body taken as the reference origin) equals the number of degrees of freedom. Then the equations are written in form (7), and, consequently, can be written in the more convenient form (6). As was proposed in [14], it proves to be convenient to write the Euler equation in some "mean" coordinate system. Then it is necessary to introduce three additional variables and to use three coupling equations. These equations must be chosen so that

$$\int [x_* \times u_*]\, dm = 0$$

Here x_* is the radius-vector of the element of mass dm in the undeformed system, u_* is the displacement vector of dm.

We introduce a vector describing the state of the system

$$x = \{\theta, \beta\}^T, \qquad \theta = \{\theta_1, \theta_2, \theta_3\}^T$$

Here θ_1, θ_2, θ_3 are the angles describing the orientation of the coordinate system, β is the vector describing the internal deformation. It can be shown [15] that the equations of motion are written in form (6), where

$$M = \begin{Vmatrix} I & 0 \\ 0 & M_d \end{Vmatrix}, \qquad I = \begin{Vmatrix} I_{11} & 0 & -J_1 \\ 0 & I_{22} & -J_2 \\ -J_1 & -J_2 & I_{33} \end{Vmatrix}$$

$$K = \omega_0^2 \begin{Vmatrix} I_{33} - I_{22} + J_3 & 0 & 0 & -A_2^T \\ 0 & I_{33} - I_{11} + J_3 & 0 & A_1^T \\ 0 & 0 & 0 & 0 \\ -A_2 & A_1 & 0 & \Pi_d \end{Vmatrix}$$

In these relations I is the system's inertia matrix. The matrices M_d, Π_d, A_α are functions of a number of parameters. It is clear that this system is not asymptotically stable because K is not positive definite. In addition, the dissipation is not complete.

The kinetic moment is constant and is directed along the axis X_3^*. Therefore, its components along the axes normal to axis X_3^* must remain zero; in addition, the magnitude of this vector must remain constant and equal to $H_0 = (I_{33} + J_3)\,\omega_0$. In the body coordinate system the components of H have the form

$$\begin{aligned} H_1 &= I_{11}\theta_1^{\cdot} - J_1\theta_3^{\cdot} - \omega_0 I_{11}\theta_2 + \omega_0 A_1^T\beta \\ H_2 &= I_{22}\theta_2^{\cdot} - J_2\theta_3^{\cdot} + \omega_0 I_{22}\theta_1 + \omega_0 A_2^T\beta \\ H_3 &= H_0 - J_1\theta_1^{\cdot} - J_2\theta_2^{\cdot} + I_{33}\theta_3^{\cdot} + \omega_0 J_2\theta_2 - \omega_0 J_2\theta_1 + \omega_0 A_3^T\beta \end{aligned}$$

Then the coupling equations are

$$H_1 + \theta_2 H_3 = 0, \qquad H_2 - \theta_1 H_3 = 0, \qquad H_3 - H_0 = 0$$

or

$$I\theta^{\cdot} + B^T\theta + A^T\beta = 0, \qquad A^T = [A_1^T A_2^T A_3^T]^T$$

$$B^T = \omega_0 \begin{Vmatrix} 0 & I_{33} - I_{11} + J_3 & 0 \\ -(I_{33} - I_{22} - J_3) & 0 & 0 \\ -J_2 & J_1 & 0 \end{Vmatrix}$$

By introducing the vector $y = \{\theta, \beta, \beta^{\cdot}\}$, we can obtain the function

$$V' = {}^1/_2\, y^T K' y$$

by adding the term $-\frac{1}{2}\theta^{\cdot T}II^{-1}\theta^{\cdot}$ to the Hamiltonian function. Matrix K' must be positive definite for asymptotic stability. Matrix K' is not positive definite if θ_3 is an ignorable coordinate. Nevertheless, it is positive definite in the other variables, which yields asymptotic stability with respect to a part of the variables.

For a nongyroscopic system $J_\alpha = 0$ and the matrix K' takes the form

$$K' = \left\| \begin{array}{ccccc} \frac{I_{33}}{I_{22}}(I_{33} - I_{22}) & 0 & 0 & -\frac{I_{33}}{I_{22}} A_2^T & 0 \\ 0 & \frac{I_{33}}{I_{11}}(I_{33} - I_{11}) & 0 & \frac{I_{33}}{I_{11}} A_1^T & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{I_{33}}{I_{22}} A_2^T & \frac{I_{33}}{I_{11}} A_1^T & 0 & \Pi_d + \sum_\alpha \frac{A_\alpha A_\alpha^T}{I_{\alpha\alpha}} & 0 \\ 0 & 0 & 0 & 0 & M_d \end{array} \right\| \qquad (15)$$

The function V' is positive definite in the variables θ_1, θ_2, β, $\beta^{\cdot}$. In the case given, the dissipation is complete; therefore, the system is asymptotically stable in these variables. It was implied above that $I_{33} > I_{22}$, $I_{33} > I_{11}$, and that the system must rotate around the axis of maximum moment of inertia. In this problem the choice of the transverse axes is arbitrary; therefore, obviously, stability is independent of I_{11} and I_{22}. The other stability conditions will depend on the system's deformation and will be determined by the rigidity of the joints.

Sometimes it is necessary to increase the system's "gyroscopic rigidity" by the introduction of an internal rotation. Such a system will include gyros. If these gyros themselves are not subject to deformation and if $J_1 = J_2 = 0$, then the system must rotate, as before, around the principal axis. In this case matrix K' is analogous to matrix (15) wherein the quantity I_{33} is replaced by $I_{33} + J_3$. The stability conditions corresponding to the rigid motion now take the form

$$(I_{33} + J_3)(I_{33} + J_3 - I_{22}) > 0, \quad (I_{33} + J_3)(I_{33} + J_3 - I_{11}) > 0$$

These conditions are necessary and can be obtained for particular models [16].

It should be noted that the arguments presented above are valid for any freely-rotating deformable system and, therefore, are valid for systems of arbitrarily coupled rigid bodies. The corresponding matrices now cannot be obtained by the method in [7] in its usual form because the presence of "loops" in the system configuration leads to the presence of couplings between the variables. Nevertheless, the general approach proposed in the given paper remains valid.

REFERENCES

1. Zajac, E. E., The Kelvin-Tait-Chetaev theorem and extensions. J. Astronaut. Sci., Vol. 11, №2, 1964.
2. Pringle, R. Jr., Stability of damped mechanical systems. AIAA Journal, Vol. 3, №2, 1965.
3. Zajac, E. E., Comments on stability of damped mechanical systems and further extensions. AIAA Journal, Vol. 3, 1965.

4. Pringle, R. Jr., On the stability of a body with connected moving parts. AIAA Journal, Vol. 4, № 8, 1966.
5. Whittaker, E. T., A treatise on the analytical dynamics of particles and rigid bodies. Cambridge Univ. Press, 1937.
6. Hooker, W. W. and Margulies, G., The dynamical attitude equations for an n-body satellite. J. Astronaut. Sci., Vol. 12, № 4, 1965.
7. Roberson, R. E. and Wittenburg, J., Proc. of the 3rd Congress Internat. Federation of Automatic Control, London, 1966.
8. Hooker, W. W., A set of r dynamical attitude equations for an arbitrary n-body satellite having r rotational degrees of freedom. AIAA Journal, Vol. 8, №7, 1970.
9. Roberson, R. E. and Likins, P. W., A linearization tool for use with matrix formalism of rotational dynamics. Ingr-Arch., Bd. 37, Nr. 6, 1969.
10. Risito, C., On the Liapunov stability of system with known first integrals. Mechanica, Vol. 2, № 4, 1967.
11. Magnus, K., Beiträge zur Dynamik des kräftefreien kardanisch gelagerten Kreisels. ZAMM, Bd. 35, Nr. 1, 1955.
12. Rumiantsev, V. V., On the stability of motion of a gyroscope on gimbals. PMM Vol. 22, № 3, 1958.
13. Leimanis, E., The general problem of the motion of coupled rigid bodies about a fixed point. N. Y., Springer-Verlag, 1965.
14. Willems, P. Y., Attitude stability of deformable satellites. Evolut. attitude et stabilis. satellite, Colloq. internat. Paris, 1968.
15. Willems, P. Y., Stability of deformable gyrostats on a circular orbit. J. Astronaut. Sci., Vol. 18, № 2, 1970.
16. Likins, P. W., Attitude stability criteria for dual-spin-spacecraft. J. Spacecraft and Rockets. Vol. 4, № 12, 1967.

Translated by N. H. C.

UDC 531.36

ON BIFURCATION AND STABILITY OF STATIONARY MOTIONS IN CERTAIN PROBLEMS OF DYNAMICS OF A SOLID BODY

PMM Vol. 38, № 4, 1974, pp. 616-627

V. N. RUBANOVSKII

(Moscow)

(Received December 19, 1973)

Bifurcation theory for stationary motions was developed by Poincaré [1] and Chetaev [2] for Lagrangian conservative mechanical systems. This theory is based on the investigation of the (transformed) potential energy of the system $V = V(c, q_1, \ldots, q_m)$, where $q_1, \ldots, q_m$ are the Lagrange coordinates and c is a parameter. For three problems in solid body dynamics we have shown below that this theory is applicable for the investigation of systems with known first integrals $U(x_1, \ldots, x_n) = c, \quad U_1(x_1, \ldots, x_n) = c_1, \ldots, U_k(x_1, \ldots, x_n) = c_k \quad (k + 1 < n)$

As in the classical case, here we can introduce the function